Precision spectroscopy in electronic and muonic H and He$^+$

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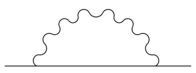
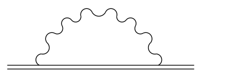
Bound–state QED

Free particle and bound–state QED

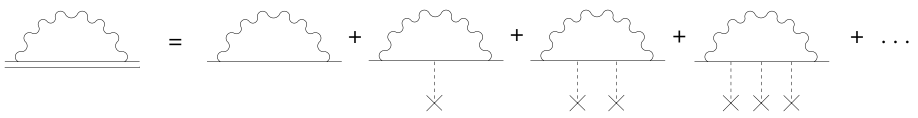
Precision QED for free particles (*e.g.*, $g-2$) is being calculated with an increasing number (several thousands) of complicated diagrams (up to the five–loop level). Bound–state QED deals with simpler diagrams (up to the two(three)–loop level), but the charged particles are bound rather than free, and thus the Coulomb exchange is not a small effect. Free QED involves only one small parameter α, while bound–state QED needs at least three expansion parameters α, $Z\alpha$ and m/M (perturbation theory):

- α, the power of which indicates the number of QED loops.
- $Z\alpha$ is the Coulomb strength. It represents the binding effect. α and $Z\alpha$ expansions behave quite differently. There is a number of contributions where we need to sum over an infinite number of Coulomb exchanges, like for the Bethe logarithm. If $Z\alpha$ is not small (Uranium $Z\alpha \approx 0.7$), there is strong coupling, and perturbation theory can not be applied. For $Z\alpha \to 0$ there is a non–analytic behaviour of the perturbation theory. The result is the occurrence of numerous logarithms ($\ln^i(1/Z\alpha)^2$) and large coefficients in the expansion.
- m/M is the recoil parameter. In the non–relativistic case the two–body system can be exactly solved by introducing the reduced mass of the system, but the separation of center–of–mass and relative motion can not be done in a relativistically covariant way. This complicates the treatment of bound–states fundamentally.

$$S_F(q) = \frac{i}{\not q - m}$$  $$S_B(q) = \frac{i}{\not q - m - \gamma^0 V}$$ 

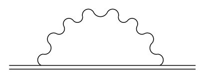
For high energy of the exchanged virtual photon, the Dirac–Coulomb propagator may be expanded as

$$\frac{1}{\not q - m - \gamma^0 V} = \frac{1}{\not q - m} + \frac{1}{\not q - m}\gamma^0 V\frac{1}{\not q - m} + \frac{1}{\not q - m}\gamma^0 V\frac{1}{\not q - m}\gamma^0 V\frac{1}{\not q - m} + \cdots \quad (V = \frac{Z\alpha}{r}, r \sim (Z\alpha))$$



One–loop self–energy in hydrogen: perturbative vs. all–order approach

$$\Delta E^{(1)}_{SE} = m\frac{\alpha}{\pi}\frac{(Z\alpha)^4}{n^3}F_n(Z\alpha)$$



- Perturbative expansion of the Dirac-Coulomb propagator in $(Z\alpha)$

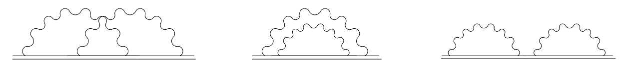
$$F_n = A_{40} + A_{41}\ln(Z\alpha)^{-2} + (Z\alpha)A_{50} + (Z\alpha)^2\left[A_{62}\ln^2(Z\alpha)^{-2} + A_{61}\ln(Z\alpha)^{-2} + G\right]$$
$$G = A_{60} + (Z\alpha)\left[A_{71}\ln(Z\alpha)^{-2} + A_{70}\right] + (Z\alpha)^2\left[A_{83}\ln^3(Z\alpha)^{-2} + A_{82}\ln^2(Z\alpha)^{-2} + A_{81}\ln(Z\alpha)^{-2} + A_{80}\right]$$

The calculation of the corrections of relative order $(Z\alpha)^2$ is highly non–trivial because the binding Coulomb fields enter in a nonperturbative way, and there is no closed–form expression for the Dirac–Coulomb propagator. For example, the separation of $(Z\alpha)^2$ relative contributions involves hundreds of terms. The uncertainty related to truncation of the $(Z\alpha)$ expansion is 28 kHz.

- All–order numerical treatment of the Dirac–Coulomb propagator leads to a self–energy with 0.8 Hz uncertainty. Both methods are in agreement.

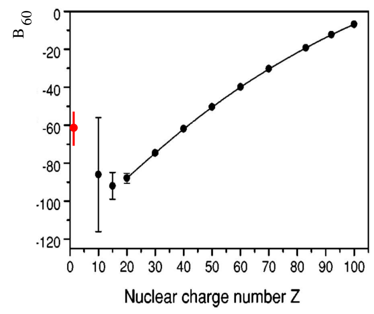
Two–loop self–energy in hydrogen: perturbative vs. all–order approach

$$\Delta E^{(2)}_{SE} = m\left(\frac{\alpha}{\pi}\right)^2\frac{(Z\alpha)^4}{n^3}G_n(Z\alpha)$$



$$G_n = B_{40} + (Z\alpha)B_{50} + (Z\alpha)^2\left[B_{63}\ln^3(Z\alpha)^{-2} + B_{62}\ln^2(Z\alpha)^{-2} + B_{61}\ln(Z\alpha)^{-2} + G_{h.o}\right] + \cdots$$

The expansion of the two–loop self–energy in powers of $Z\alpha$ and $\ln[(Z\alpha)^{-2}]$ leads to surprisingly large terms and is therefore considered a prototype for badly converging series. Bad convergence of the $(Z\alpha)$ expansion and disagreement between the perturbative and nonperturbative approach require progress in this field, both from the theoretical and experimental side.



Hydrogen and muonic hydrogen

Energy levels and definition of Lamb shifts

The atomic energy levels are described by

$$E = \frac{1}{n^2}R_\infty\frac{m_r}{m} + L(\alpha, c, R_p\ldots)$$

where R_∞ is the Rydberg constant, m_r the reduced mass of the system, m the electron mass, α the fine structure constant, R_p the proton rms charge radius, and L the Lamb shift.
The Lamb shift is defined as any deviation of the energy level from the prediction of the Dirac (Schrödinger) equation caused by radiative (QED), recoil and nuclear structure corrections.
Note that in order to predict the hydrogen energy levels within bound–state QED, we need to determine, with proper accuracy, fundamental constants like $R_\infty, \alpha, \hbar, m\ldots$ and R_p.

High precision spectroscopy in hydrogen

Several transition frequencies have been measured with high accuracy (*e.g.*, H($1S-2S$) $\delta\nu/\nu \approx 10^{-14}$) in hydrogen and deuterium

$$\left.\begin{array}{lcl} \Delta E_{1S-2S} & = & 0.75\,R_\infty\frac{m_r}{m} - L_{1S} + L_{2S} \\ \Delta E_{2S-8S/D} & = & 0.23\,R_\infty\frac{m_r}{m} - L_{2S} + L_{8S/D} \\ \ldots & & \\ L_{nS} & = & \frac{1}{n^3}L_{1S} + \varepsilon_n \end{array}\right\} \Rightarrow \left\{\begin{array}{lcr} L^{\rm exp}_{1S} & = & 8172.840(22)\ \text{MHz} \\ R_\infty & = & 3\,289\,841\,960.360(22)\ \text{MHz} \end{array}\right.$$

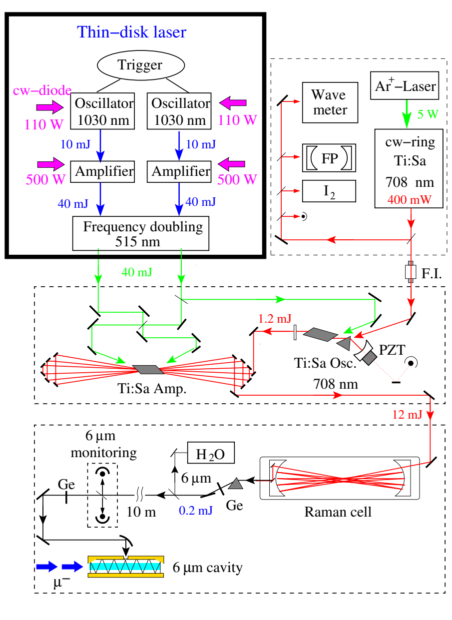
and L and R_∞ have been extracted. To reach such a precison the lasers used to excite the transitions were referenced to an atomic Cs-clock which defines the second.

Comparison of measured and calculated 1S Lamb shift in H, and the proton radius

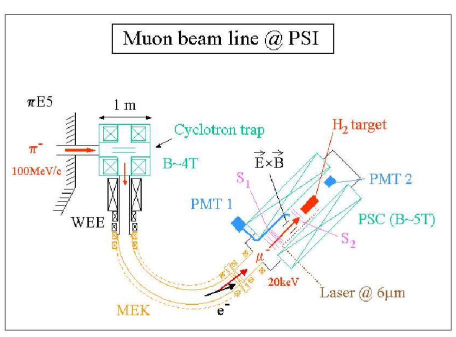
$$\left.\begin{array}{r} R_p \\ \alpha,\ c,\ \hbar,\ m\ldots \\ \text{bound-state QED} \end{array}\right\} \Rightarrow \begin{array}{ll} L^{\rm th}_{1S} = 8172.901(4)(51)\ \text{MHz} & R_p \text{ from e–p scattering with 2\% accuracy} \\ L^{\rm th}_{1S} = 8172.901(4)(3)\ \text{MHz} & R_p \text{ from } \mu p \text{ Lamb shift with } 10^{-3} \text{ accuracy} \end{array}$$

The comparison between theory and experiment is presently limited by the uncertainty of the proton radius to a level of 6×10^{-6}. A measurement of R_p by the muonic hydrogen Lamb shift experiment opens the way to check bound-state QED to a level of 3×10^{-7}.

Muonic hydrogen Lamb shift experiment at PSI



An experiment using a new ultra–low energy muon beam (bottom) and a dedicated $\lambda = 6\ \mu$m laser system (left) was set up at PSI, and first data were taken in Nov. 2003. The statistics were too low to determine the $2S-2P$ resonance line. Next data taking is planned for 2007, with a thin–disk laser developed at IFSW Stuttgart, enhancing the event rate by a factor 20.



Helium

A very challenging experiment is underway at MPQ aiming to measure the $1S-2S$ transition frequency in He$^+$. A He$^+$ single ion is cooled, trapped and directly two–photon excited with a newly invented frequency comb in the XUV region ($\lambda = 60$ nm).
Since bound–state QED corrections to the energy levels scale like $Z^{\geq4}$, spectroscopy on He$^+$ offers the possibility to test the interesting QED corrections $\sim$16 times better than in H.

Contributions	H($1S-2S$)	He$^+$($1S-2S$)
$\Delta E_{\rm Dirac} \approx \Delta E_{\rm Bohr} = \frac{3}{4}Z^2R_\infty$	2.5×10^{12} kHz	1×10^{12} kHz
Uncertainty related to R_∞	(16 kHz)	(65 kHz)
Near future uncertainty of R_∞ ($1S-3S$ spectroscopy in H)	($\sim$ 8 kHz)	($\sim$ 30 kHz)
Radiative and recoil contributions (main contribution scales like $Z^{3.7}$)	7 126 785 kHz	93 794 104 kHz
QED uncertainty given by the B_{60} and B_{7i} terms, scaling with $Z^{\geq6}$	(3 kHz)	(185 kHz)
Uncalculated higher terms ($C_{50}\ldots$, recoil)	(2 kHz)	(100 kHz)
Nuclear finite size: $\sim (Z\alpha)^4 m_r^3\ R^2_{\rm p/He}\ \delta_{l0}$	1 102 kHz	62 005 kHz
Uncertainty using CODATA values	(44 kHz)	(369 kHz)
Near Future: R_p ($R_{\rm He}$) from muonic hydrogen (helium) experiments	(2 kHz)	(40 kHz)
Calculated nuclear polarizability	-0.08 kHz	-28 kHz
Uncertainty	(0.02 kHz)	(6 kHz)

As can be seen from the table He$^+$ is less sensitive to R_∞ and more sensitive to the problematic ($(Z\alpha)^6$) bound–state QED terms than H.
As in the case of H, the uncertainty resulting from the nuclear size term has to be reduced by an order of magnitude in order to compare QED predictions with the measurements. Measuring the $2S-2P$ energy difference in μ^4He$^+$ ions to 50 ppm will result in a determination of the ^{4}He nuclear radius to 3×10^{-4}, good enough to exploit the potential of the He$^+$($1S-2S$) spectroscopy experiment.

Muonic Helium

Spectroscopy on muonic He$^+$ offers the way to extract the nuclear radius with high precision.
QED, recoil and nuclear structure corrections contribute to the muonic helium Lamb shift as

$$\Delta E_{2S-2P}(\mu^4\text{He}) = (1814.7(6) - 102.6R^2_{\rm He}[\text{fm}^2])\ \text{meV} = 1814.7(6) - 289.6(1.7) = 1525.1(1.8)\ \text{meV}$$

A measurement of $\Delta E_{2S-2P}(\mu^4\text{He})$ with 50 ppm accuracy will lead to a determination of the He$^+$ nuclear radius 10 times better than presently known (5% relative accuracy of muonic helium polarizability required).

The μHe experiment can be performed at PSI, with small changes of the setup presently used for the μp experiment.
The μHe experiment is significantly easier compared to muonic hydrogen since

	μHe	μp	
• $2S$ lifetime	1.4 μs at 30 mbar He	1.0 μs at 1 mbar H_2	shorter target
• $2S$ population	2.5%	1% longlived	
• Two–photon $2S-1S$ decay	measurable	non–measurable	monitoring of the $2S$ pop.
• $2P-1S$ transition energy	8.2 keV	1.9 keV	detector less delicate
• Laser pulse	E=5–10 mJ at $\lambda = 812$ nm	E=0.2 mJ at $\lambda = 6\ \mu$m	

The laser light required for the μHe experiment at 812 nm can be produced with the existing TiSa laser optically pumped by our thin–disk laser developed at IFSW Stuttgart.

Conclusion

- Bound–state QED involves a rich spectrum of problems, and deserves to be tested, particularly because of the lack of well established universal prescriptions appropriate for the relativistic two–body problem in general. Combining precision spectroscopy in H and He$^+$ will lead to sensitive tests of the perturbative and nonperturbative methods used for bound–state QED. Determination of the p and ^{4}He nuclear sizes by measuring muonic Lamb shifts is absolutely essential.
- Bound–state QED plays an important role in the determination of fundamental constants, like R_∞ ($R_\infty = \alpha^2mc/2h$), the electron mass m (bound–state g-factor of hydrogenlike ions and antiprotonic He), m_μ/m and hence μ_μ/μ_p (hyperfine splitting in muonium), α (bound–state g-factor experiments if the theory can be carried out to the 10^{-12} level), m/M where M is the proton mass (hydrogen energy levels).
- H, D, ^{3}He, ^{4}He nuclear radii can be determined by spectroscopy with the same apparatus used for the μp Lamb shift. This is of relevance for precision spectroscopy and nuclear physics.